Proving partial correctness of `while`-programs

Lecture #6.2

Notations used in verification

- Assertions to be proved in program verification:
  - Statements of mathematics:

    $$(x + 1)^2 = x^2 + 2x + 1$$

  - Partial correctness specifications:

    $$\{P\}\ C\ \{Q\}$$

  - Total correctness specifications

    $$[P]\ C\ [Q]$$

Basic terms in formal logic

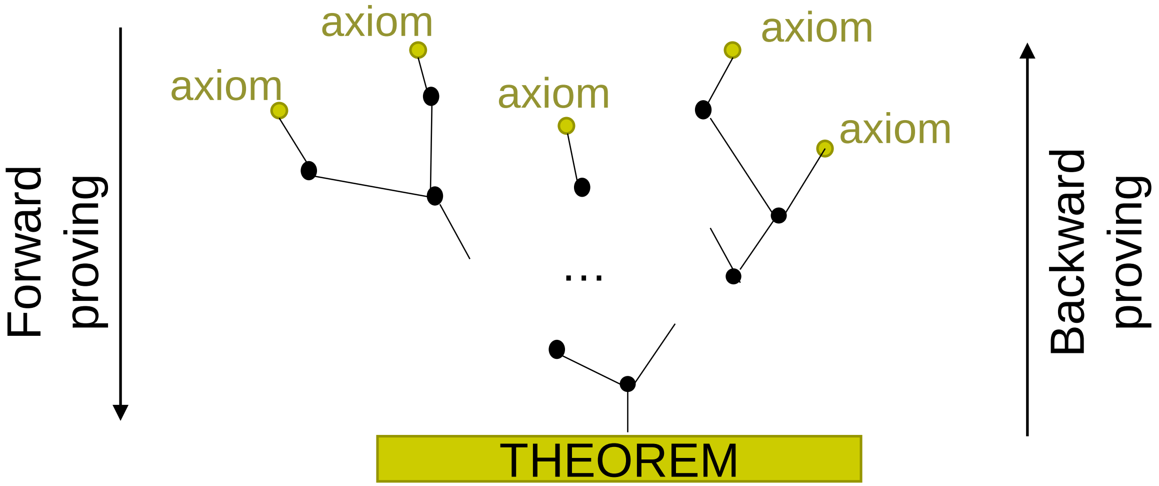
- Floyd-Hoare logic (FHL) gives rules for proving the partial and total correctness of programs, i.e. terms $\vdash \{P\}\ C\ \{Q\}$ and $\vdash [P]\ C\ [Q]$
- Predicate calculus gives rules for proving theorems of predicate logic
- Arithmetics gives decision rules for proving statements about numbers
- Theorems are statements, which can be ***proved*** to be true or false.
- Axioms are statements which are ***assumed*** to be true.
- $\vdash S$ means that S can be proved (unconditionally) using proof rules.
- $\Gamma \vdash S$ means that S can be deduced from the axioms $\Gamma = \{A_1, A_2, \ldots, A_n\}$ using proof rules.

Elements of proof theory

- *Deduction* (*proof*) - sequence (tree) of *statements* where every statement is either
  - an *axiom* *or*
  - deduced from true statements by *proof rules*
- Properties of the proof rules:
  - *Correctness* (*soundness*) - it is not possible to deduce something that is not correct from correct assumptions.
  - *Completeness* - all statements that are valid are deducible from axioms using the proof rules.
- *Deduction system* $\cong$ *set of axioms* (*or axiom schemas*) + *set of inference rules*

Proof

- Typically the proof has a shape of a tree where
  - theorem is the root of the tree and
  - axioms are leaves.
  - The edges correspond to applications of inference rules



Inference rule

- *The inference rule* is an instruction on how to make a proof step
- The rules may have differenct form. The rules of FHL are of the form

$$\frac{\overbrace{\vdash S_1, \ldots, \vdash S_n}^{\text{hypothesis}}}{\underbrace{\vdash S}_{\text{conclusion}}}$$

- This means the conclusion $\vdash S$ may be deduced from the hypotheses $\vdash S_1, \ldots, \vdash S_n$
- The hypotheses can either be theorems of FHL or predicate calculus
- Example: (a rule of propositional logic**)**

$$\frac{\vdash p \Rightarrow q \qquad\qquad \vdash q \Rightarrow p}{\vdash p \Leftrightarrow q}$$

Hoare type proof systems

- There is an axiom or inference rule for “elimination“ of <u>programming language constructs </u>such as *if*, *while* etc. from HFL formulae.
- This allows to simplify the triples in backward proofs.
- Instead of concrete axioms FHL uses axiom schemas that are instantiated by concrete program conditions.
- The order of applying inference rules in the proof is determined by the ***syntactic structure*** of the program to be verified.
- This makes constructing proofs much easier.

The rules of FHL: sequential composition

- Syntax:

$$\frac{\vdash \{P\}\ C1\ \{Q\}, \qquad \vdash \{Q\}\ C2\ \{R\},}{\vdash \{P\}\ C1\ ;\ C2\ \{R\}}$$

- Semantics:

  If triples $\{P\}C1\{Q\}$ and $\{Q\}C2\{R\}$, have proofs, then also triple that includes sequential composition of $C1$ and $C2$, i.e. $\{P\}\ C1; C2\ \{R\}$ has a proof.

- Example:

$$\frac{\vdash \{X=1\}\ \texttt{X:=X+1}\ \{X=2\} \qquad \vdash \{X=2\}\ \texttt{X:=X+1}\ \{X=3\}}{\vdash \{X=1\}\ \texttt{X:=X+1; X:=X+1}\ \{X=3\}}$$

FHL rules: sequential composition example

Assume we have given tripples 1-3:

1. $\vdash \{X = x \wedge Y = y\}\ R := X\ \{R = x \wedge Y = y\}$
2. $\vdash \{R = x \wedge Y = y\}\ X := Y\ \{R = x \wedge X = y\}$
3. $\vdash \{R = x \wedge X = y\}\ Y := R\ \{Y = x \wedge X = y\}$

by sequential coposition rule (1.) and (2.) provide

4. $\vdash \{X = x \wedge Y = y\}\ R := X;\ X := Y\ \{R = x \wedge X = y\}$

and (4.), (3.) yield

5. $\vdash \{X = x \wedge Y = y\}\ R := X;\ X := Y;\ Y := R\ \{Y = x \wedge X = y\}$

FHL rules: SKIP-*axiom*

- Syntax:

$$\vdash \{P\} \ \texttt{SKIP} \ \{P\}$$

- Semantics:
  - Program state does not change when skip is executed
- Explanation:
  - This is axiom scheme where P may be any assertion
- Examples of concrete SKIP-axioms:
  - $\vdash \{Y = 2\} \ \texttt{SKIP} \ \{\mathrm{Y} = 2\}$
  - $\vdash \{\mathrm{T}\} \ \texttt{SKIP} \ \{\mathrm{T}\}$
  - $\vdash \{Y = K \times X + R\} \ \texttt{SKIP} \ \{Y = K \times X + R\}$

FHL rules: assignment

- Syntax: $V := E$
- Semantics:
  The state is changed by assigning the value of the term E to the variable V. All other variables preserve their values
- Example: Y:=Y+5

  This adds 5 to the value of the variable Y.

- Variable substitution:
  - $P[E/V]$ denotes the result of replacing all occurrences of V in P by E.
- Example: $(X + 1 > X)\,[Y + Z / X] \;=\; ((Y + Z) + 1 > Y + Z)$
- Following property holds: $V[E/V] = E$

FHL: assignment axiom:

$\vdash \{P[E/V]\}$ `V:=E` $\{P\}$

where

- *V* is variable, *E* is an expression, *P* is any statement
- $P[E/V]$ – denotes the result of substituting the term E for all occurrences of the variable V in statement P.

- Explanation:
  - the value of a variable `V` after executiong an assignment `V:=E`
  - equals the value of the expression `E` in the state before executing it.

- Example:
  - $\vdash \{Y = 2\}$ `X:=2` $\{Y = X\}$
  - $\vdash \{Z = X^Y\}$ `X:=X**Y` $\{Z = X\}$

FHL rules: precondition strenghtening

$$\frac{\vdash P \Rightarrow P' \qquad\qquad \vdash \{P'\}\ C\ \{Q\}}{\vdash \{P\}\ C\ \{Q\}}$$

- Application example
  From implication $\vdash X = n \Rightarrow X + 1 = n + 1$
  and
  assignment axiom $\vdash \{X + 1 = n + 1\}\ X := X + 1\ \{X = n + 1\}$
  we can deduce

  $$\vdash \{X = n\}\ X := X + 1\ \{X = n + 1\},$$

  where n is auxilliary variable that occurs only in pre-and post-conditions .

FHL rules: postcondition weakening

$$\frac{\vdash \{P\}\ C\ \{Q'\} \qquad \vdash Q' \Rightarrow Q}{\vdash \{P\}\ C\ \{Q\}}$$

- Example (application of rules in forward reasoning):

Proof step	Used inference rule
1. $\vdash \{R = X \wedge 0=0\}\ Q:= 0\ \{R=X \wedge Q=0\}$	(assignment axiom)
2. $\vdash R = X \Rightarrow R = X \wedge 0 = 0$	(logic equality
3. $\vdash \{R = X\}\ Q := 0\ \{R = X \wedge Q = 0\}$	(precondition strengthening)
4. $\vdash R = X \wedge Q = 0 \Rightarrow R = X + (Y \times Q)$	(arithmetic equality)
5. $\vdash \{R = X\}\ Q := 0\ \{R = X + (Y \times Q)\}$	(postcondition weakening)

FHL rules: `BEGIN-END` -blocks

- Syntax:

  `BEGIN VAR` $V1$; … ; Vn ; C `END`

- Semantics:
  - Variables $V1$; … ; Vn are used locally within the block. After C is executed the values of $V1$; … ; Vn are restored to the values they had befor the block was entered
  - The initial values for $V1$; … ; Vn in the block are unspecified.

- Example:

  `BEGIN VAR R; R := X; X:=Y; Y:=R END`

  - Variables `X` and `Y` exchange their values by using an auxiliary variable `R`.

FHL rules: `BEGIN-END` -blocks

Block-rule**:**

$$\frac{\vdash \{P\}\ \texttt{C}\ \{Q\}}{\vdash \{P\}\ \texttt{BEGIN VAR V1; … ;Vn ; C END}\ \{Q\}}$$

where none of the variables $V1;\ \ldots\ ;Vn$ occur in P or Q.

Explanation:

This restriction of variable occurrence in P and Q is because their value is determined locally only within the block. Their valuation outside the block may be different.

FHL rules: `IF`- command

- Syntax:

  $\{P\}$ `IF S  THEN C1  ELSE C2` $\{Q\}$

- Semantics:
  - If the statement `S` is *true* (in current state), then `C1` is executed
  - If `S` *is false* then `C2` is executed

- Example:
  - `IF X < Y  THEN  MAX:= Y  ELSE MAX:= X`

FHL rules: `IF`- command

- <u>`IF` - rule1 (one branch):</u>

$$\frac{\vdash \{P \wedge S\}\ \mathtt{C}\ \{Q\} \qquad \vdash P \wedge \neg S \Rightarrow Q}{\vdash \{P\}\ \mathtt{IF\ S\ THEN\ \ C}\ \{Q\}}$$

- <u>`IF`- rule 2 (two branches):</u>

$$\frac{\vdash \{P \wedge S\}\ \mathtt{C}_1\ \{Q\}, \qquad \vdash \{P \wedge \neg S\}\ \mathtt{C}_2\ \{Q\}}{\vdash \{P\}\ \mathtt{IF\ S\ THEN\ \ C}_1\ \mathtt{ELSE\ C}_2\ \{Q\}}$$

FHL rules: application example of `IF`- command

- Example:

Given

- $\vdash \{T \wedge \quad X >= Y\}$ `MAX := X` $\{MAX = max(X, Y)\}$
- $\vdash \{T \wedge \neg(X >= Y)\}$ `MAX := Y` $\{MAX = max(X, Y)\}$

By `IF`-rule 2 it follows:

- $\vdash \{T\}$ `IF X>=Y THEN MAX:=X ELSE  MAX:=Y` $\{MAX= max(X, Y)\}$

FHL rules: `WHILE`-command

- Syntax: `WHILE` *S* `DO` *C*

- Semantics:
  - If the statement *S* is true in the current state, then *C* is executed and the `WHILE` command is repeated.
  - If *S* is false, then exit the command,
  - Command *C* is repeatedly executed until the value of *S* becomes *false*
  - If *S* never becomes *false*, then the execution never terminates

- Example:

  `WHILE ¬(X = 0) DO  X := X – 2`

For which values of X the command does not terminate?

FHL rules: `WHILE`-command

- <u>Invariants</u>

  Suppose

  $$\vdash \{P \wedge S\}\ C\ \{P\}$$

  then P is an invariant of C whenever S holds.

- <u>Explanation</u> (`WHILE`-rule):
  - if the execution of `WHILE`-command body C preserves truth value of P *once*, then it preserves this truth value for arbitrary number of executions of C.
  - If `WHILE`-command has terminated, then loop condition S must be $false$ (because this is `WHILE` termination condition).

FHL rules: `WHILE`-command

- <u>(Simple) WHILE–rule:</u>

$$\frac{\vdash \{P \wedge S\}\ \mathtt{C}\ \{P\}}{\vdash \{P\}\ \mathtt{WHILE}\ S\ \mathtt{DO}\ \mathtt{C}\ \{P \wedge \neg S\}}$$

- <u>Extended WHILE-rule:</u>

$$\frac{\vdash P \Rightarrow R \qquad \vdash \{R \wedge S\}\ \mathtt{C}\ \{R\} \qquad \vdash R \wedge \neg S \Rightarrow Q}{\vdash \{P\}\ \mathtt{WHILE}\ S\ \mathtt{DO}\ \mathtt{C}\ \{Q\}}$$

`WHILE`-command: invariant

How to find an invariant?

- It must hold initially when entering the loop
- With negated test it must establish the result of loop
- The body must leave it unchanged

- Intuition:
  - The invariant says that what *has been done* so far together with what remains *to be done* gives the *desired result*.

  - Analogy with milestone where one face indicates the distance passed and the other the distance to go, their sum is the total distance between the departure point and destination.

`WHILE`-command: example 1

- Example (factorial program 1):

$\{X = n \wedge Y = 1\}$

```
  WHILE X≠0 DO
  BEGIN Y:=Y×X; X:=X-1  END
```

$\{X = 0 \wedge Y = n!\}$

- Analyze the variable values
  - Finally `X = 0` and `Y = n!`
  - Initially `X = n` and `Y = 1`
  - On each loop Y is increased and X is decreased.

WHILE-command: example 1

- How the variables keep their values in execution?
  - Y holds the result so far
  - X! is what remains to be computed
  - n! is the desired result

→ The invariant is $X! \times Y = n!$

`WHILE`-command: example 2

- Example (factorial program 2):

$$\{X = n \wedge Y = 1\}$$

```
  WHILE X<N DO
  BEGIN X:=X+1; Y:=Y×X; END
```

$$\{Y = N!\}$$

- Analyze the variable values
  - Finally `X = N` and `Y = N!`
  - Initially `X = 0` and `Y = 1`
  - On each loop both `X` and `Y` increase.

`WHILE`-command: example 2

```
{X=0 ∧ Y=1}
 WHILE X<N DO
   BEGIN X:=X+1; Y:=Y×X END
{Y=N!}
```

- How the values of variables evolve in execution?
  - At end `Y = N!`
  - and $\neg(X<N) \Rightarrow X=N$
  - The invariant must be

$$Y = X! \wedge X \leq N$$

FHL rules: conjunction and disjunction of specifications

$$\frac{\vdash \{P_1\}\ C\ \{Q_1\} \qquad\qquad \vdash \{P_2\}\ C\ \{Q_2\}}{\vdash \{P_1 \wedge P_2\}\ C\ \{Q_1 \wedge Q_2\}}$$

$$\frac{\vdash \{P_1\}\ C\ \{Q_1\} \qquad\qquad \vdash \{P_2\}\ C\ \{Q_2\}}{\vdash \{P_1 \vee P_2\}\ C\ \{Q_1 \vee Q_2\}}$$

- These rules allow splitting large triples into simpler ones and prove them separately. To prove

$$\vdash \{P_1 \wedge P_2\}\ C\ \{Q_1 \wedge Q_2\}.$$

- it sufices proving independently

$$\vdash \{P_1\}\ C\ \{Q_1\} \quad \text{and} \quad \vdash \{P_2\}\ C\ \{Q_2\}$$

FHL rules: FOR-command

- Syntax:

```
FOR V := E1 UNTIL  E2  DO C
```

  - Restriction: index variable V must not occur in E1 or E2 or be the left hand side of an assignment in C.

- Semantics:
  - If the values of terms *E*1 and *E*2 are positive numbers $e1$ and $e2$, where $e1 \leq e2$, then C is executed $(e2 - e1) + 1$ times
  - with the variable *V* taking values $e1$, $e1+1$, $e1+2$, … , $e2$.
  - for any other value of *V* the FOR-command acts as `skip`.

- Example:

```
FOR  N:=1 UNTIL  M  DO X:=X+N
```

  - expressions E1 and E2 are evaluated only once at the entry to FOR-command;
  - if E1 and E2 do not have positive integer value or E1>E2, then FOR-command does nothing.

Reduction to WHILE-command

- FOR-command

  $\texttt{FOR V:=E}_1\ \texttt{UNTIL E}_2\ \texttt{DO C}$

  is equivalent to WHILE-program

  BEGIN VAR V;

  $\quad \texttt{V:=E}_1\texttt{;}$

  $\quad \texttt{WHILE V} \geq \texttt{E}_1 \wedge \texttt{V} \leq \texttt{E}_2\ \texttt{DO}$

  $\quad\quad$ BEGIN

  $\quad\quad\quad$ C;

  $\quad\quad\quad \texttt{V:=}V\texttt{+1}$

  $\quad\quad$ END

  END

Annotating the FOR-command

- Having an annotated FOR-command

$$\{P\}\ \texttt{FOR V:= }E_1\texttt{ UNTIL }E_2\texttt{ DO }\{R\}\ \texttt{C}\ \{Q\}$$

- we can transform it to equivalent annotated WHILE program
- Invariant *R* of this WHILE program must include condition $V \leq E_2+1$

$\{P\}$

```
BEGIN VAR V;
   V:= E1;
   WHILE V≥E1 ∧ V≤E2 DO {R} BEGIN
        C;
        V:=V+1
   END
END
```

$\{Q\}$

FHL rules: FOR-command

- FOR–axiom:

$$\vdash \{R \wedge (E2 < E1)\}\ \texttt{FOR}\ V\texttt{:=E1 UNTIL E2 DO C}\ \{R\}$$

- (Primary) FOR–rule:

$$\frac{\vdash \{R \wedge (E1 \leq V) \wedge (V \leq E2)\}\ \texttt{C}\ \{R[V+1/V]\}}{\vdash \{R[E1/V]\} \wedge (E1 \leq E2)\}\ \texttt{FOR}\ V\texttt{:=E1 UNTIL E2 DO C}\ \{R[E2+1/V]\},}$$

- Extended FOR–rule:

$$\frac{\vdash P \Rightarrow R[E1/V] \qquad \vdash R[E2+1/V] \Rightarrow Q \qquad \vdash P \wedge (E2 < E1) \Rightarrow Q \qquad \vdash \{R \wedge (E1 \leq V) \wedge (V \leq E2)\}\ \texttt{C}\ \{R[V+1/V]\}}{\vdash \{P\}\ \texttt{FOR V:=E1 UNTIL E2 DO}\ \{R\}\ \texttt{C}\ \{Q\}}$$

Summary

- The proof system must be *sound* and *complete,*
- i.e. it is necessary to show that axioms are valid and inference rules imply true conclusions if their hypothesis are true.
- The calculus is complete if all its valid assertions are provable.
- FHL is *relative complete* if for all programs of the FHL triples expressible in it can be transformed to programming construct free logic formuli.
- Ed. Clarke proved that for languages that include <u>*recursion*</u>, <u>*static scoping*</u>, <u>*global variables*</u> and <u>*parametrized procedure calls*</u>, there does not exist sound and complete FHL.