The Landau Symbols

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1 Limits

Definition 1 (Finite limit)**.** If $(x_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers and $x \in [0, \infty)$, then $\lim_{n\to\infty} x_n = x$ if and only if

$$\forall \varepsilon \in (0, \infty) \,.\, \exists n \in \mathbb{N} \,.\, \forall m > n \,.\, x - \varepsilon < x_m < x + \varepsilon \quad .$$

Definition 2 (Infinite limit)**.** If $(x_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers, then $\lim_{n\to\infty} x_n = \infty$ if and only if

$$\forall y \in [0, \infty) \,.\, \exists n \in \mathbb{N} \,.\, \forall m > n \,.\, x_m > y \quad .$$

Definition 3 (Finite limit inferior and superior)**.** For every sequence $(x_n)_{n\in\mathbb{N}}$ of nonnegative real numbers, we define the following:

- If the set

  $$\{\, x \in [0, \infty) \mid \forall \varepsilon \in (0, \infty) \,.\, \exists n \in \mathbb{N} \,.\, \forall m > n \,.\, x - \varepsilon < x_m \,\}$$

  has a maximum x, then $\liminf_{n\to\infty} x_n = x$.

- If the set

  $$\{\, x \in [0, \infty) \mid \forall \varepsilon \in (0, \infty) \,.\, \exists n \in \mathbb{N} \,.\, \forall m > n \,.\, x_m < x + \varepsilon \,\}$$

  has a minimum x, then $\limsup_{n\to\infty} x_n = x$.

Definition 4 (Infinite limit inferior and superior)**.** If $(x_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative real numbers, then $\liminf_{n\to\infty} x_n = \infty$ if and only if

$$\forall y \in [0, \infty) \,.\, \exists n \in \mathbb{N} \,.\, \forall m > n \,.\, x_m > y \quad ,$$

and $\limsup_{n\to\infty} x_n = \infty$ if and only if

$$\forall y \in [0, \infty) \,.\, \forall n \in \mathbb{N} \,.\, \exists m > n \,.\, x_m > y \quad .$$

Lemma 1 (Basic properties of limit inferior and superior). *If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of non-negative real numbers, then* $\liminf_{n\to\infty} x_n$ *and* $\limsup_{n\to\infty} x_n$ *exist, and* $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Lemma 2 (Relationship to ordinary limit). *If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of nonnegative real numbers, then the following holds:*

- *If* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, *then* (x_n) *has a limit.*
- *If* (x_n) *has a limit, then* $\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Lemma 3 (Limit superior and ordering). *If* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are sequences of nonnegative real numbers with* $x_n \leq y_n$ *for all* $n \in \mathbb{N}$, *then* $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$.

Lemma 4 (Limit superior of sums and products). *If* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are sequences of nonnegative real numbers with* $\limsup_{n\to\infty} x_n < \infty$ *and* $\limsup_{n\to\infty} y_n < \infty$, *then the following holds:*

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$
$$\limsup_{n\to\infty} x_n y_n \leq \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right)$$

Lemma 5 (Limit inferior and superior of reciprocals). *If* $(x_n)_{n\in\mathbb{N}}$ *is a sequence of positive real numbers, then the following holds:*

$$\liminf_{n\to\infty} x_n = 0 \Longleftrightarrow \limsup_{n\to\infty} x_n^{-1} = \infty$$
$$\limsup_{n\to\infty} x_n = 0 \Longleftrightarrow \liminf_{n\to\infty} x_n^{-1} = \infty$$

2 The Landau Symbols

Definition 5 (Landau symbols). The functions[1] $O, o, \Omega, \omega, \Theta : (0,\infty)^{\mathbb{N}} \to \mathscr{P}\left((0,\infty)^{\mathbb{N}}\right)$ are defined as follows:

$$O(f) = \left\{ g : \mathbb{N} \to (0,\infty) \;\middle|\; \limsup_{n\to\infty} \frac{g(n)}{f(n)} < \infty \right\}$$
$$o(f) = \left\{ g : \mathbb{N} \to (0,\infty) \;\middle|\; \limsup_{n\to\infty} \frac{g(n)}{f(n)} = 0 \right\}$$
$$\Omega(f) = \left\{ g : \mathbb{N} \to (0,\infty) \;\middle|\; \liminf_{n\to\infty} \frac{g(n)}{f(n)} > 0 \right\}$$
$$\omega(f) = \left\{ g : \mathbb{N} \to (0,\infty) \;\middle|\; \liminf_{n\to\infty} \frac{g(n)}{f(n)} = \infty \right\}$$
$$\Theta(f) = \left\{ g : \mathbb{N} \to (0,\infty) \;\middle|\; 0 < \liminf_{n\to\infty} \frac{g(n)}{f(n)} \leq \limsup_{n\to\infty} \frac{g(n)}{f(n)} < \infty \right\}$$

[1]Note that for sets A and B, the notation B^A means $\{f \mid f : A \to B\}$.

Note that in number theory, Ω is defined differently, namely as follows:

$$\Omega(f) = \left\{ g : \mathbb{N} \to (0, \infty) \;\middle|\; \limsup_{n\to\infty} \frac{g(n)}{f(n)} > 0 \right\}$$

Lemma 6 (Relationships between Landau functions). *For all functions $f, g : \mathbb{N} \to (0, \infty)$, the following holds:*

$$\begin{array}{cccc} g \in O(f) \Leftrightarrow f \in \Omega(g) & o(f) \subset O(f) & O(f) \cap \omega(f) = \emptyset & O(f) \cap \Omega(f) = \Theta(f) \\ g \in o(f) \Leftrightarrow f \in \omega(g) & \omega(f) \subset \Omega(f) & \Omega(f) \cap o(f) = \emptyset & \end{array}$$